Chapter 3

Probability and simulation (solutions to exercises)

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Import Python packages

```
# Import all needed python packages
import numpy as np
import scipy.stats as stats
import matplotlib.pyplot as plt
import statsmodels.api as sm
import statsmodels.stats.power as smp
```

3.1 Concrete items

|||| Exercise 3.1 Concrete items

A construction company receives concrete items for a construction. The length of the items are assumed reasonably normally distributed. The following requirements for the length of the elements are made

$$\mu = 3000 \text{ mm}.$$

The company samples 9 items from a delevery which are then measured for control. The following measurements (in mm) are found:

3003	3005	2997	3006	2999	2998	3007	3005	3001

a) Compute the following three statistics: the sample mean, the sample standard deviation and the standard error of the mean, and what are the interpretations of these statistics?

b) In a construction process, 5 concrete items are joined together to a single construction with a length which is then the complete length of the 5 concrete items. It is very important that the length of this new construction is within 15 m plus/minus 1 cm. How often will it happen that such a construction will be more than 1 cm away from the 15 m target (assume that the population mean concrete item length is $\mu = 3000$ mm and that the population standard deviation is $\sigma = 3$)?

c) Find the 95% confidence interval for the mean μ.

d) Find the 99% confidence interval for μ. Compare with the 95% one from above and explain why it is smaller/larger!

e) Find the 95% confidence intervals for the variance σ^2 and the standard deviation σ.

f) Find the 99% confidence intervals for the variance σ^2 and the standard deviation σ.

3.2 Aluminum profile

Exercise 3.2 Aluminum profile

The length of an aluminum profile is checked by taking a sample of 16 items whose length is measured. The measurement results from this sample are listed below, all measurements are in mm:

180.02	180.00	180.01	179.97	179.92	180.05	179.94	180.10
180.24	180.12	180.13	180.22	179.96	180.10	179.96	180.06

From data is obtained: $\bar{x} = 180.05$ and $s = 0.0959$.

It can be assumed that the sample comes from a population which is normal distributed.

a) A 90%-confidence interval for μ becomes?

b) A 99%-confidence interval for σ becomes?

3.3 Concrete items (hypothesis testing)

|||| Exercise 3.3 Concrete items (hypothesis testing)

This is a continuation of Exercise 1, so the same setting and data is used (read the initial text of it).

a) To investigate whether the requirement to the mean is fulfilled (with $\alpha = 5\%$), the following hypothesis should be tested

$$H_0 : \mu = 3000$$
$$H_1 : \mu \neq 3000.$$

Or similarly asked: what is the evidence against the null hypothesis?

b) What would the level $\alpha = 0.01$ critical values be for this test, and what are the interpretation of these?

c) What would the level $\alpha = 0.05$ critical values be for this test (compare also with the values found in the previous question)?

d) Investigate, by some plots, whether the data here appears to be coming from a normal distribution (as assumed until now)?

e) Assuming that you, maybe among different plots, also did the normal q-q plot above, the question is now: What exactly is plotted in that plot? Or more specifically: what are the x- and y-coordinates of e.g. the two points to the lower left in this plot?

3.4 Aluminum profile (hypothesis testing)

|||| Exercise 3.4 Aluminium profile (hypothesis testing)

We use the same setting and data as in Exercise 2, so read the initial text of it.

a) Find the evidence against the following hypothesis:

$$H_0 : \mu = 180.$$

b) If the following hypothesis test is carried out

$$H_0 : \mu = 180,$$
$$H_1 : \mu \neq 180.$$

What are the level $\alpha = 1\%$ critical values for this test?

c) What is the 99%-confidence interval for μ?

d) Carry out the following hypothesis test

$$H_0 : \mu = 180,$$
$$H_1 : \mu \neq 180,$$

using $\alpha = 5\%$.

3.5 Transport times

|||| Exercise 3.5 Transport times

A company, MM, selling items online wants to compare the transport times for two transport firms for delivery of the goods. To compare the two companies recordings of delivery times on a specific route were made, with a sample size of n = 9 for each firm. The following data were found:

$$\text{Firm A: } \bar{y}_A = 1.93 \text{ d and } s_A = 0.45 \text{ d,}$$
$$\text{Firm B: } \bar{y}_B = 1.49 \text{ d and } s_B = 0.58 \text{ d.}$$

note that d is the SI unit for days. It is assumed that data can be regarded as stemming from normal distributions.

a) We want to test the following hypothesis

$$H_0 : \mu_A = \mu_B$$
$$H_1 : \mu_A \neq \mu_B$$

What is the p-value, interpretation and conclusion for this test (at $\alpha = 5\%$ level)?

b) Find the 95% confidence interval for the mean difference $\mu_A - \mu_B$.

c) What is the power of a study with $n = 9$ observations in each of the two samples of detecting a potential mean difference of 0.4 between the firms (assume that $\sigma = 0.5$ and that we use $\alpha = 0.05$)?

d) What effect size (mean difference) could be detected with $n = 9$ observations in each of the two samples with a power of 0.8 (assume that $\sigma = 0.5$ and that we use $\alpha = 0.05$)?

e) How large a sample size (from each firm) would be needed in a new investigation, if we want to detect a potential mean difference of 0.4 between the firms with probability 0.90, that is with `power=0.90` (assume that $\sigma = 0.5$ and that we use $\alpha = 0.05$)?

3.6 Cholesterol

|||| Exercise 3.6 Cholesterol

In a clinical trial of a cholesterol-lowering agent, 15 patients' cholesterol (in mmol/L) has been measured before treatment and 3 weeks after starting treatment. Data are listed in the following table:

Patient	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Before	9.1	8.0	7.7	10.0	9.6	7.9	9.0	7.1	8.3	9.6	8.2	9.2	7.3	8.5	9.5
After	8.2	6.4	6.6	8.5	8.0	5.8	7.8	7.2	6.7	9.8	7.1	7.7	6.0	6.6	8.4

The following is run in Python:

```
x1 = np.array([9.1, 8.0, 7.7, 10.0, 9.6, 7.9, 9.0, 7.1,
        8.3, 9.6, 8.2, 9.2, 7.3, 8.5, 9.5])
x2 = np.array([8.2, 6.4, 6.6, 8.5, 8.0, 5.8, 7.8, 7.2,
        6.7, 9.8, 7.1, 7.7, 6.0, 6.6, 8.4])

# Independent
t_obs,p_val = stats.ttest_ind(x1,x2,equal_var=False)
print(t_obs, p_val)

3.3206371615181602 0.0025776851099507973


# Paired
t_obs,p_val = stats.ttest_rel(x1,x2)
print(t_obs,p_val)

7.340653674886342 3.6722518645935146e-06
```

a) Can there, based on these data be demonstrated a significant decrease in cholesterol levels with $\alpha = 0.001$?

3.7 Pulse

|||| Exercise 3.7 Pulse

13 runners had their pulse measured at the end of a workout and 1 minute after again and we got the following pulse measurements:

Runner	1	2	3	4	5	6	7	8	9	10	11	12	13
Pulse end	173	175	174	183	181	180	170	182	188	178	181	183	185
Pulse 1min	120	115	122	123	125	140	108	133	134	121	130	126	128

The following was run in Python:

```
Pulse_end = np.array([173,175,174,183,181,180,170,
                      182,188,178,181,183,185])
Pulse_1min = np.array([120,115,122,123,125,140,108,
                       133,134,121,130,126,128])
print(np.mean(Pulse_end))

179.46153846153845


print(np.mean(Pulse_1min))

125.0


print(np.std(Pulse_end, ddof=1))

5.189980485117207


print(np.std(Pulse_1min, ddof=1))

8.406346808612328


print(np.std(Pulse_end-Pulse_1min, ddof=1))

5.767948575466911
```

a) What is the 99% confidence interval for the mean pulse drop (meaning the drop during 1 minute from end of workout)?

b) Consider now the 13 pulse end measurements (first row in the table). What is the 95% confidence interval for the standard deviation of these?

3.8 Foil production

|||| Exercise 3.8 Foil production

In the production of a certain foil (film), the foil is controlled by measuring the thickness of the foil in a number of points distributed over the width of the foil. The production is considered stable if the mean of the difference between the maximum and minimum measurements does not exceed 0.35 mm. At a given day, the following random samples are observed for 10 foils:

Foil	1	2	3	4	5	6	7	8	9	10
Max. in mm (y_{max})	2.62	2.71	2.18	2.25	2.72	2.34	2.63	1.86	2.84	2.93
Min. in mm (y_{min})	2.14	2.39	1.86	1.92	2.33	2.00	2.25	1.50	2.27	2.37
Max-Min (D)	0.48	0.32	0.32	0.33	0.39	0.34	0.38	0.36	0.57	0.56

The following statistics may potentially be used

$$\bar{y}_{\text{max}} = 2.508,\ \bar{y}_{\text{min}} = 2.103,\ s_{y_{\text{max}}} = 0.3373,\ s_{y_{\text{min}}} = 0.2834,\ s_D = 0.09664.$$

a) What is a 95% confidence interval for the mean difference?

b) How much evidence is there that the mean difference is different from 0.35? State the null hypothesis, t-statistic and p-value for this question.

3.9 Course project

|||| Exercise 3.9 Course project

At a specific education it was decided to introduce a project, running through the course period, as a part of the grade point evaluation. In order to assess whether it has changed the percentage of students passing the course, the following data was collected:

	Before introduction of project	After introduction of project
Number of students evaluated	50	24
Number of students failed	13	3
Average grade point $\bar{x}$	6.420	7.375
Sample standard deviation s	2.205	1.813

a) As it is assumed that the grades are approximately normally distributed in each group, the following hypothesis is tested:

$$H_0 : \mu_{\text{Before}} = \mu_{\text{After}},$$
$$H_1 : \mu_{\text{Before}} \neq \mu_{\text{After}}.$$

The test statistic, the p-value and the conclusion for this test become?

b) A 99% confidence interval for the mean grade point difference is?

c) A 95% confidence interval for the grade point standard deviation after the introduction of the project becomes?

3.10 Concrete items (sample size)

|||| Exercise 3.10 Concrete items (sample size)

This is a continuation of Exercise 1, so the same setting and data is used (read the initial text of it).

a) A study is planned of a new supplier. It is expected that the standard deviation will be approximately 3, that is, $\sigma = 3$ mm. We want a 90% confidence interval for the mean value in this new study to have a width of 2 mm. How many items should be sampled to achieve this?

b) Answer the sample size question above but requiring the 99% confidence interval to have the (same) width of 2 mm.

c) (Warning: This is a difficult question about a challenging abstraction - do not worry, if you do not make this one) For the two sample sizes found in the two previous questions find the probability that the corresponding confidence interval in the future study will actually be more than 10% wider than planned for (still assuming and using that the population variance is $\sigma^2 = 9$).

d) Now a new experiment is to be planned. In the first part above, given some wanted margin of error (ME) a sample size of $n = 25$ was found. What are each of the probabilities that an experiment with $n = 25$ will detect effects corresponding to ("end up significant for") $\mu_1 = 3001, 3002, 3003$ respectively? Assume that we use the typical $\alpha = 0.05$ level and that $\sigma = 3$?

e) One of the sample size computation above led to $n = 60$ (it is not so important how/why). Answer the same question as above using $n = 60$.

f) What sample size would be needed to achieve a power of 0.80 for an effect of size 0.5?

g) Assume that you only have the finances to do an experiment with $n = 50$. How large a difference would you be able to detect with probability 0.8 (i.e. Power$=0.80$)?

3.11 Concrete items 2

|||| Exercise 3.11 Concrete items 2

A construction company receives concrete items for a construction. The length of the items are assumed reasonably normally distributed, with mean $\mu = 3000$ mm and standard deviation $\sigma = 3$ mm.

a) In a construction process, 5 concrete items are joined together to a single construction with a length which is then the complete length of the 5 concrete items. It is very important that the length of this new construction is within 15 m plus/minus 2 cm. How often will it happen that such a construction will be more than 2 cm away from the 15 m target (assume that the population mean concrete item length is $\mu = 3000$ mm and that the population standard deviation is $\sigma = 3$)?

The construction company, wants to check that the delivered concrete items do in fact have an average length of 3000 mm and a standard deviation of 3 mm (still assuming that the lengths follow a normal distribution). The company samples 9 items from a delivery which are then measured for control. The following measurements (in mm) are found:

3003	3005	2997	3006	2999	2998	3007	3005	3001

b) Compute the following three statistics: the sample mean, the sample standard deviation and the standard error of the mean. Explain the difference between the sample standard deviation and the standard error of the mean.

c) Find the 95% confidence interval for the mean μ.

d) Find the 99% confidence interval for μ. Compare with the 95% one from above and explain why it is smaller/larger!

e) Find the 95% confidence intervals for the variance σ^2 and the standard deviation σ.

f) Based on the 95% confidence intervals, what should the construction company conclude about the delivered concrete items? Can they trust that the lengths of the individual items have mean 3000 mm and standard deviation 3 mm?

g) What would be the conclusion (based on 95% confidence intervals) if similar sample estimates had been based on a much larger sample? Say the construction company makes a sample of $n = 100$ concrete items and from this sample they obtain $\bar{x} = 3002.3$ and $s = 3.7$.

3.12 Concrete items 2 (hypothesis testing)

|||| Exercise 3.12 Concrete items (hypothesis testing)

This is a continuation of Exercise 11:

A construction company receives concrete items for a construction. The length of the items are assumed reasonably normally distributed, with mean $\mu = 3000$ mm and standard deviation $\sigma = 3$ mm.

The company samples 9 items from a delivery which are then measured for control. The following measurements (in mm) are found:

3003	3005	2997	3006	2999	2998	3007	3005	3001

a) To investigate whether the requirement to the mean is fulfilled (with $\alpha = 5\%$), the following hypothesis should be tested

$$H_0 : \mu = 3000$$
$$H_1 : \mu \neq 3000.$$

Or similarly asked: what is the evidence against the null hypothesis?

b) At significance level $\alpha = 0.01$, what are the critical values for t_{obs}? And what is the interpretation of these critical values?

c) At significance level $\alpha = 0.05$, what are the critical values for t_{obs}? And what is the interpretation of these critical values (compare also with the values found in the previous question)?

d) The company would now like to check the assumption that the data follows a normal distribution. Plot a histogram of the data and compare to a normal distribution pdf with same mean and standard deviation. Also plot the ecdf and compare to a normal distribution cdf with same mean and standard deviation (ecdf/cdf comparison has the advantage that no histogram-binsize is needed).

e) Plot a QQ-plot of the data (see Method 3.42). Do you think the normal assumption is valid?

f) Take a closer look at your QQ-plot. What is actually plotted on the y-axis? And what is plotted on the x-axis?

g) In order to evaluate the QQ-plot visually, plot QQ-plots of 9 simulated random samples from a normal distribution (with sample size $n = 9$ as in the data). Then replace one of the plots (e.g, the middle one) with the QQ-plot of the real data and evaluate visually. Does the QQ-plot of the real data look "as good" as the simulated ones?

You can use the following code:

```
# Enter data into variable x:
x = np.array([3003,3005,2997,3006,2999,2998,3007,3005,3001])

# Generate 9 plots
fig, axs = plt.subplots(3, 3, figsize=(10, 10))

# Create 9 QQ plots of random normal samples
for ax in axs.flat:
    sm.qqplot(stats.norm.rvs(size=9), line="q", ax=ax)
    ax.set_ylim([-3, 3])

# Replace middle plot with real data
axs[1,1].clear()
sm.qqplot((x - x.mean())/x.std(ddof=1), line="q", ax=axs[1,1])
axs[1,1].set_ylim([-3, 3])

# Highlight middle plot with red spines
plt.setp(axs[1,1].spines.values(), color="red")

# Generate the plot
plt.tight_layout()
plt.show()
```