Chapter 9

The general linear model

Exercises

Contents

9.1 Proof of Theorem 9.14

|||| Exercise 9.1 Proof of Theorem 9.14

a) Prove Theorem 9.14, using the definition in equation (9-38).

9.2 Independence and correlation

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a) Simulate Y_1, X and Y_2 using the setting in Example 9.15.

b) Check that both Y_1 and Y_2 are normal, and plot Y_2 as a function of Y_1.

c) Calculate the correlation between Y_1 and Y_2 and plot Y_2 as a function of Y_1 and comment on the results

9.3 Proff of Eq. (9-50)

|||| Exercise 9.3 Proff of Eq. (9-50)

a) Prove that rowsums of $\boldsymbol{A}$ in (9-49) is equal zero, i.e. that $\boldsymbol{A}\mathbf{1} = \mathbf{0}$

b) Prove Eq. (9-48)

c) Prove Eq. (9-50).

9.4 Proof of Corollary 9.18

|||| Exercise 9.4 Proff of Corollary 9.18

a) Show that when $\boldsymbol{Y} \sim N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ then, $\boldsymbol{Z} = \boldsymbol{\Lambda}^{-1/2}\boldsymbol{V}^T(\boldsymbol{Y} - \boldsymbol{\mu}) \sim N_n(\boldsymbol{0}, \boldsymbol{I})$, with $\boldsymbol{V}$, and $\boldsymbol{\Lambda}$ as in Lemma 9.3.

b) Prove Corollary 9.18.

9.5 Projection matrix

|||| Exercise 9.5 Projection matrix

a) Use exercise 3 to show that $\boldsymbol{A}$ in (9-49) is an orthogonal projection matrix.

9.6 Proof of Lemma 9.22

|||| Exercise 9.6 Proof of Lemma 9.22

a) Use Lemma 9.3, property 1 of Lemma 9.22 and Theorem 9.5 to prove property 2 of Lemma 9.22.

9.7 Correlation

|||| Exercise 9.7 Correlation

a) With $\boldsymbol{r}$ as in (9-51) what is the correlation between r_i and r_j?

9.8 Lag-1 autocorrelation

|||| Exercise 9.8 Lag-1 autocorrelation

Consider the random variables $\epsilon_i \sim N(0, \sigma^2)$, iid. and $t = \{1, ..., n\}$. Now consider the correlation estimate,

$$\hat{\rho}_\epsilon(1) = \frac{\sum_{t=1}^{n-1} \epsilon_t \epsilon_{t+1}}{\sum_{t=1}^{n} \epsilon_t^2} = \frac{C}{Q}, \tag{9-1}$$

the idea of the questions below is that show that $\hat{\rho}_\epsilon(1) \approx N(0, 1/n)$ by showing that $V[\hat{\rho}_\epsilon(1)] \approx 1/n$. $\hat{\rho}_\epsilon(1)$ is simpler than $\hat{\rho}(1)$ in (9-129), but for n large the behavior is similar.

a) Show that $E[C] = 0$, $E[Q] = n\sigma^2$, $V[C] = (n-1)\sigma^4$, $V[Q] = 2n\sigma^4$, and $Cov[C, Q] = 0$.

b) Use the result from question a) and non-linear error propagation to show that $V[\hat{\rho}_\epsilon(1)] \approx 1/n$, for n large.

9.9 Orthogonal projections

|||| Exercise 9.9 Orthogonal projections

a) With $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$ as in (9-70), show that $Cov[\boldsymbol{H}_1\boldsymbol{Z}, \boldsymbol{H}_2\boldsymbol{Z}] = \boldsymbol{0}$. Hint: Use Theorem 9.10 and Exercise 5.

9.10 Proof of Corollary 9.29

|||| Exercise 9.10 Proof of Corollary 9.29

In this exercise we will prove Corollary 9.29 by a series of sub questions.

a) Show that if $\boldsymbol{Y} \sim N(\boldsymbol{X}\boldsymbol{\beta}, \sigma^2 \boldsymbol{I})$ then

$$\boldsymbol{Y}^T(\boldsymbol{I} - \boldsymbol{H}_1)\boldsymbol{Y} \sim \chi^2(n-2). \tag{9-2}$$

Independently of the value of $\boldsymbol{\beta}$

b) Show that if $\boldsymbol{Y} \sim N(\mathbf{1}\mu, \sigma^2 \boldsymbol{I})$ then

$$\boldsymbol{Y}^T(\boldsymbol{H}_1 - \boldsymbol{H}_0)\boldsymbol{Y} \sim \chi^2(1). \tag{9-3}$$

independently of the value of μ, you may use the the formulation in (9-137) to calculate $\boldsymbol{H}_1$, or simply use the fact that $\mathbf{1}\boldsymbol{H}_1 = \mathbf{1}^T$ (see Exercise 11).

c) Show that if $\boldsymbol{Y} \sim N(\mathbf{0}, \sigma^2 \boldsymbol{I})$ then

$$\boldsymbol{Y}^T\boldsymbol{H}_0\boldsymbol{Y} \sim \chi^2(1). \tag{9-4}$$

9.11 t-test Orthogonal projections

|||| Exercise 9.11 t-test Orthogonal projections

a) Show that the projection matrices in rhs of (9-141) are orthogonal i.e. $\boldsymbol{H}_0(\boldsymbol{H}_1 - \boldsymbol{H}_0) = \mathbf{0}$, $\boldsymbol{H}_0(\boldsymbol{I} - \boldsymbol{H}_1) = \mathbf{0}$ and $(\boldsymbol{H}_1 - \boldsymbol{H}_0)(\boldsymbol{I} - \boldsymbol{H}_1) = \mathbf{0}$. Hint: you may start by showing that $\boldsymbol{X}_0^T\boldsymbol{H}_1 = \boldsymbol{X}_0^T$. You may use the parametrization (9-137).

b) Use the result to show that

$$\begin{aligned} Cov[\boldsymbol{H}_0\boldsymbol{Y}, (\boldsymbol{H}_1 - \boldsymbol{H}_0)\boldsymbol{Y}] &= \boldsymbol{0} \\ Cov[\boldsymbol{H}_0\boldsymbol{Y}, (\boldsymbol{I} - \boldsymbol{H}_1)\boldsymbol{Y}] &= \boldsymbol{0} \\ Cov[(\boldsymbol{H}_1 - \boldsymbol{H}_0)\boldsymbol{Y}, (\boldsymbol{I} - \boldsymbol{H}_1)\boldsymbol{Y}] &= \boldsymbol{0} \end{aligned} \tag{9-5}$$

and hence that the projected vectors are independent. Also what is the interpretation in trems of fitted values?

9.12 t-test $\hat{\sigma}^2$ central

|||| Exercise 9.12 t-test $\hat{\sigma}^2$ central

a) Show that $\hat{\sigma}^2$ (in Equation (9-147)) is a central estimator for the variance in the LM, and find $V[\hat{\sigma}^2]$.

9.13 t-test Central estmators under Null-hypothesis

|||| Exercise 9.13 t-test Central estmators under Null-hypothesis

Consider the projection matrices for the two sample t-test (equation (9-141)), consider two groups $Y_{1,i} \sim N(\mu_1, \sigma^2)$ and iid., $i = \{1, 2, ..., n_1\}$ and $Y_{2,j} \sim N(\mu_2, \sigma^2)$ and iid., $j = \{1, 2, ..., n_2\}$. Define $\boldsymbol{Y} = [\boldsymbol{Y}_1^T, \boldsymbol{Y}_2^T]^T = [Y_{1,1}, ..., Y_{1,n_1}, Y_{2,1}, ..., Y_{1,n_2}]^T$ and

a) Show that

$$\boldsymbol{Y}^T(\boldsymbol{H}_1 - \boldsymbol{H}_0)\boldsymbol{Y} = \frac{n_1 n_2}{n_1 + n_2}(\bar{Y}_1 - \bar{Y}_2)^2 \tag{9-6}$$

b) Show that $E[\boldsymbol{Y}^T(\boldsymbol{H}_1 - \boldsymbol{H}_0)\boldsymbol{Y}] = \frac{n_1 n_2}{n_1+n_2}(\mu_1 - \mu_2)^2 + \sigma^2$

- Under the assumption $\mu_1 = \mu_2 = \mu$ conclude that $\boldsymbol{Y}^T(\boldsymbol{H}_1 - \boldsymbol{H}_0)\boldsymbol{Y}$ is a central estimator for σ^2, find the variance of this estimator, and compare with the estimator (9-147).

9.14 Nested projections

|||| Exercise 9.14 Nested projections

Let $\boldsymbol{X}_i$ be as in (9-150), i.e.

$$\boldsymbol{X}_i = [\boldsymbol{X}_{i-1} \quad \tilde{\boldsymbol{X}}_i] \tag{9-7}$$

and condider the projection matrices based on $\boldsymbol{X}_{i-1} \in \mathbb{R}^{n \times p_{i-1}}$, and $\boldsymbol{X}_i \in \mathbb{R}^{n \times (p_i + q_i)}$ $(q_i > 0)$

$$\begin{aligned} \boldsymbol{H}_{i-1} &= \boldsymbol{X}_{i-1}(\boldsymbol{X}_{i-1}^T \boldsymbol{X}_{i-1})^{-1} \boldsymbol{X}_{i-1}^T \\ \boldsymbol{H}_i &= \boldsymbol{X}_i (\boldsymbol{X}_i^T \boldsymbol{X}_i)^{-1} \boldsymbol{X}_i^T \end{aligned} \tag{9-8}$$

a) Show that $\boldsymbol{X}_i^T \boldsymbol{H}_i = \boldsymbol{X}_i^T$.

b) Set $\boldsymbol{A} = (\boldsymbol{X}_i^T \boldsymbol{X}_i)^{-1}$, with

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{bmatrix} \tag{9-9}$$

with $\boldsymbol{A}_{11} \in \mathbb{R}^{p_i \times p_i}$, $\boldsymbol{A}_{12} = \boldsymbol{A}_{21}^T \in \mathbb{R}^{p_i \times q_i}$, and $\boldsymbol{A}_{22} \in \mathbb{R}^{q_i \times q_i}$, show that $\boldsymbol{A}_{kl}$ solve the equations

$$\begin{aligned} \boldsymbol{X}_{i-1}^T \boldsymbol{X}_{i-1} \boldsymbol{A}_{11} + \boldsymbol{X}_{i-1}^T \tilde{\boldsymbol{X}}_i \boldsymbol{A}_{21} &= \boldsymbol{I} \\ \boldsymbol{X}_{i-1}^T \boldsymbol{X}_{i-1} \boldsymbol{A}_{12} + \boldsymbol{X}_{i-1}^T \tilde{\boldsymbol{X}}_i \boldsymbol{A}_{22} &= \boldsymbol{0} \\ \tilde{\boldsymbol{X}}_i^T \boldsymbol{X}_{i-1} \boldsymbol{A}_{11} + \tilde{\boldsymbol{X}}_i^T \tilde{\boldsymbol{X}}_i \boldsymbol{A}_{21} &= \boldsymbol{0} \\ \tilde{\boldsymbol{X}}_i^T \boldsymbol{X}_i \boldsymbol{A}_{12} + \tilde{\boldsymbol{X}}_i^T \tilde{\boldsymbol{X}}_i \boldsymbol{A}_{22} &= \boldsymbol{I} \end{aligned} \tag{9-10}$$

c) Use the result above to show that $\boldsymbol{X}_{i-1}^T \boldsymbol{H}_i = \boldsymbol{X}_{i-1}^T$.

9.15 t-test parametrization

|||| Exercise 9.15 t-test parametrization

a) Assuming that $Y_{1,i} \sim N(\mu_1, \sigma^2)$ and $Y_{2,j} \sim N(\mu_2, \sigma^2)$ are iid and $i \in \{1, ..., n_1\}$ and $j \in \{1, ..., n_2\}$ formulate an LM (i.e. parametrize $\boldsymbol{X}$)

$$\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}; \quad \boldsymbol{\epsilon} \sim N(\boldsymbol{0}, \sigma^2 \boldsymbol{I}, \tag{9-11}$$

with

$$\begin{bmatrix} \boldsymbol{Y}_1 \\ \boldsymbol{Y}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{1}_{n_1} & a\boldsymbol{1}_{n_1} \\ \boldsymbol{1}_{n_2} & b\boldsymbol{1}_{n_2} \end{bmatrix} \tag{9-12}$$

such that the parametrization is orthogonal and $\hat{\beta}_1 = \frac{1}{n_1+n_2}(n_1\bar{Y}_1 + n_2\bar{Y}_2)$, i.e. the average of all observation, and $\hat{\beta}_2 = \bar{Y}_1 - \bar{Y}_2$.

9.16 An ill conditioned problem

|||| Exercise 9.16 An ill conditioned problem

a) Using the data from Example 9.43 fit parameters for the full model and parameter for a reduced model and compare the parameters values.

9.17 Helmert transformation

|||| Exercise 9.17 Helmert transformation

a) With reference to (9-243) show that

$$T_{HN} = \begin{bmatrix} 1 & -1/2 & -1/3 & -1/4 & \dots & -1/k \\ 1 & 1/2 & -1/3 & -1/4 & \dots & -1/k \\ 1 & 0 & 2/3 & -1/4 & \dots & -1/k \\ 1 & 0 & 0 & 3/4 & \dots & -1/ \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ 1 & 0 & \dots & & 0 & (k-1)/k \end{bmatrix}, \tag{9-13}$$

b) Using $\boldsymbol{X}$ as in (9-234) show that

$$\boldsymbol{X}\boldsymbol{T}_{HN} = \begin{bmatrix} \mathbf{1} & -\frac{1}{2}\mathbf{1} & -\frac{1}{3}\mathbf{1} & \dots & -\frac{1}{k}\mathbf{1} \\ \mathbf{1} & \frac{1}{2}\mathbf{1} & -\frac{1}{3}\mathbf{1} & \dots & -\frac{1}{k}\mathbf{1} \\ \mathbf{1} & \mathbf{0} & \frac{2}{3}\mathbf{1} & \dots & -\frac{1}{k}\mathbf{1} \\ \vdots & & & \ddots & \vdots \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \dots & \frac{k-1}{k}\mathbf{1} \end{bmatrix} \tag{9-14}$$

c) Show that

$$(\boldsymbol{X}_{HN}^T\boldsymbol{X}_{HN})^{-1} = \frac{1}{n}\begin{bmatrix} \frac{1}{k} & 0 & 0 & \dots & 0 \\ 0 & 2 & 0 & \dots & 0 \\ 0 & 0 & \frac{3}{2} & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \frac{k}{k-1} \end{bmatrix} \tag{9-15}$$

d) Use the above to prove (9-244).

9.18 Paired t-test

|||| Exercise 9.18 Paired t-test

a) Show that the parametrization in (9-259) is an orthogonal parametrization.

b) Find the parameter estimates based on the desing matrix (9-259).

c) Find the projecion matrix corresponding to the desing matrix (9-259).

d) Prove (9-262) (Hint: you may use that $\boldsymbol{Y}_i^T \boldsymbol{E} = n\bar{Y}_i \mathbf{1}^T$)

9.19 2-way Anova sum-constraint

|||| Exercise 9.19 2-way Anova sum-constraint

a) Find a matrix $\boldsymbol{T}$ such that

$$\tilde{\boldsymbol{\beta}} = \boldsymbol{T}\boldsymbol{\beta} \tag{9-16}$$

with $\tilde{\boldsymbol{\beta}} = [\mu, \alpha_1, ...\alpha_k, \beta_1, ..., \beta_l]^T$ and $\boldsymbol{\beta} = [\mu, \alpha_1, ...\alpha_{k-1}, \beta_1, ..., \beta_{l-1}]^T$, such that the constraints (9-270) are fulfilled.

b) Show that the constraints (9-270) can be realized by the desing matrix in (9-271) (hint use the transfromation matrix $\boldsymbol{T}$ and the appropriate (non identifiable) desing matrix corresponding to $\tilde{\boldsymbol{\beta}}$).

9.20 Two-way ANOVA*

|||| Exercise 9.20 Two-way ANOVA*

This porpuse of this exercise is to show equation (9-278), this will rely on Kronecker products, and hence solving the exercise require basic understanding of those.

First note that the (non-unique) design matrices can be written in terms of Kronecker products as

$$\boldsymbol{X} = \begin{bmatrix} \mathbf{0}_{b,k-1} & \boldsymbol{I}_b \\ \boldsymbol{I}_{k-1} \otimes \mathbf{1}_b & \mathbf{1}_{k-1} \otimes \boldsymbol{I}_b \end{bmatrix}; \quad \boldsymbol{X}_{Tr} = \boldsymbol{I}_k \otimes \mathbf{1}_b; \quad \boldsymbol{X}_{Bl} = \mathbf{1}_k \otimes \boldsymbol{I}_b; \quad \boldsymbol{X}_0 = \mathbf{1}_k \otimes \mathbf{1}_b$$

and

a) Use the above to write the projection matrices $\boldsymbol{H}_0$, $\boldsymbol{H}_{Tr}$ and $\boldsymbol{H}_{Bl}$ in terms of Kronecker products.

b) Using (9-277) it is staight forward to show that

$$(\boldsymbol{X}^T\boldsymbol{X})^{-1} = \boldsymbol{C}_1 + \boldsymbol{C}_2 - \boldsymbol{C}_3 \tag{9-17}$$

with

$$\begin{aligned} \boldsymbol{C}_1 &= \frac{1}{b}\begin{bmatrix} \boldsymbol{I} + \boldsymbol{E}_{k-1,k-1} & -\boldsymbol{E}_{k-1,b} \\ -\boldsymbol{E}_{b,k-1} & \boldsymbol{E}_{bb} \end{bmatrix}; \quad \boldsymbol{C}_2 = \begin{bmatrix} \mathbf{0} & \mathbf{0}_{k-1,b} \\ \mathbf{0}_{b,k-1} & \frac{1}{k}\boldsymbol{I} \end{bmatrix}; \\ \boldsymbol{C}_3 &= \begin{bmatrix} \mathbf{0}_{k-1,k-1} & \mathbf{0}_{k-1,b} \\ \mathbf{0}_{b,k-1} & \frac{1}{bk}\boldsymbol{E}_{bb} \end{bmatrix}, \end{aligned} \tag{9-18}$$

show that $\boldsymbol{X}\boldsymbol{C}_1\boldsymbol{X}^T = \boldsymbol{H}_{Tr}$, $\boldsymbol{X}\boldsymbol{C}_2\boldsymbol{X}^T = \boldsymbol{H}_{Bl}$, and $\boldsymbol{X}\boldsymbol{C}_3\boldsymbol{X}^T = \boldsymbol{H}_0$, and hence that $\boldsymbol{H} = \boldsymbol{H}_{Tr} + \boldsymbol{H}_{Bl} - \boldsymbol{H}_0$.

c) Use the above to conclude that $\boldsymbol{H} - \boldsymbol{H}_{Tr} = \boldsymbol{H}_{Bl} - \boldsymbol{H}_0$ and $\boldsymbol{H} - \boldsymbol{H}_{Bl} = \boldsymbol{H}_{Tr} - \boldsymbol{H}_0$.