*Written examination*: 22. june 2017

*Course name and number*: **Introduction to Mathematical Statistics (02403)**

*Aids and facilities allowed:* All

The questions were answered by

______________ ______________ ______

(student number) (signature) (table number)

There are 30 questions of the "multiple choice" type included in this exam divided on 7 exercises. To answer the questions you need to fill in the prepared 30-question multiple choice form (on 6 seperate pages) in CampusNet

5 points are given for a correct answer and $-1$ point is given for a wrong answer. ONLY the following 5 answer options are valid: 1, 2, 3, 4 or 5. If a question is left blank or another answer is given, then it does not count (i.e. "0 points"). Hence, if more than one answer option is given to a single question, which in fact is technically possible in the online system, it will not count (i.e. "0 points"). The number of points corresponding to specific marks or needed to pass the examination is ultimately determined during censoring.

**The final answers should be given in the exam module in CampusNet. The table sheet here is ONLY to be used as an "emergency" alternative (remember to provide your study number if you hand in the sheet).**

| **Exercise** | I.1 | I.2 | I.3 | I.4 | I.5 | II.1 | III.1 | III.2 | III.3 | III.4 |
|---|---|---|---|---|---|---|---|---|---|---|
| **Question** | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) |
| **Answer** | 4 | 5 | 1 | 2 | 4 | 5 | 4 | 2 | 5 | 1 |

| **Exercise** | III.5 | IV.1 | IV.2 | V.1 | V.2 | V.3 | V.4 | V.5 | VI.1 | VI.2 |
|---|---|---|---|---|---|---|---|---|---|---|
| **Question** | (11) | (12) | (13) | (14) | (15) | (16) | (17) | (18) | (19) | (20) |
| **Answer** | 1 | 1 | 1 | 5 | 4 | 3 | 2 | 1 | 4 | 4 |

| **Exercise** | VI.3 | VI.4 | VI.5 | VI.6 | VI.7 | VI.8 | VI.9 | VI.10 | VII.1 | VII.2 |
|---|---|---|---|---|---|---|---|---|---|---|
| **Question** | (21) | (22) | (23) | (24) | (25) | (26) | (27) | (28) | (29) | (30) |
| **Answer** | 3 | 4 | 1 | 5 | 1 | 1 | 1 | 2 | 1 | 2 |

The questionnaire contains 41 pages.

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**Multiple choice questions:** *Note that not all the suggested answers are necessarily meaningful. In fact, some of them are very wrong but under all circumstances there is one and only one correct answer to each question.*

**Exercise I**

In a double-blinded placebo-controlled clinical trial, 100 subjects with high long-term blood sugar were given a new type of medicine (termed 'active'). An additional 100 subjects were given placebo. The subjects received the medicine over 26 weeks and their long-term blood sugar was determined at the start of the trial (week 0) and after the last dose was given (week 26). The following table indicates average and standard deviation for long-term blood sugar, which is measured as the concentration of the substance HbA1c [%] in the blood.

| Medicine | Week | Average | Standard dev. |
|---|---|---|---|
| Active | 0 | 8.5 | 1.1 |
| Placebo | 0 | 8.6 | 1.2 |
| Active | 26 | 7.2 | 1.4 |
| Placebo | 26 | 8.1 | 1.5 |

**Question I.1 (1)**

A 95% confidence interval for the average long-term blood sugar after 26 weeks in the group receiving the new drug (active) is:

1 ☐ $7.2 \pm t_{0.95} \cdot \frac{1.4}{\sqrt{100}}$, where the $t$-distribution with 99 degrees of freedom is used

2 ☐ $(8.5 - 7.2) \pm t_{0.975} \cdot \frac{1.4}{10}$, where the $t$-distribution with 99 degrees of freedom is used

3 ☐ $7.2 \pm t_{0.975} \cdot \frac{1.4}{\sqrt{99}}$, where the $t$-distribution with 99 degrees of freedom is used

4* ☐ $7.2 \pm t_{0.975} \cdot 0.14$, where the $t$-distribution with 99 degrees of freedom is used

5 ☐ $(8.1 - 7.2) \pm t_{0.975} \cdot \frac{1.4}{10}$, where the $t$-distribution with 99 degrees of freedom is used

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The 95% confidence interval for long term blood sugar is

$$\bar{x} \pm t_{1-\alpha/2} \frac{s}{\sqrt{n}}$$

in our case we have $\bar{x} = 7.2$, $\alpha = 0.05$, $s = 1.4$, the degrees of freedom for the t-distribution is 100-1=99, $n = 100$, and $1 - \alpha/2 = 0.975$ hence we can write the solution as

$$7.2 \pm t_{0.975} \frac{1.4}{\sqrt{100}} = 7.2 \pm t_{0.975} 0.14$$

which is answer no 4

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**Question I.2 (2)**

When the difference in the starting level is taken into account, the $p$-value for the test, of whether the effect (decrease in HbA1c) over 26 weeks of the new drug (active) differs from the placebo, can be found with the following R code (here `active_week26`, `active_week0`, `placebo_week26`, and `placebo_week0`, indicates vectors with the HbA1c levels for the individual subjects in the active group in week 26 and week 0, and in the placebo group at week 26 and 0, all sorted by subject ID):

1 ☐ `t.test(aktiv_week26, placebo_week26)`

2 ☐ `t.test(aktiv_week26, placebo_week26, paired=TRUE)`

3 ☐ `t.test(aktiv_week26 - mean(c(placebo_week0, aktiv_week0)), placebo_week26 - mean(c(placebo_week0, aktiv_week0)))`

4 ☐ `t.test(aktiv_week26 - aktiv_week0, placebo_week26 - placebo_week0, paired=TRUE)`

5* ☐ `t.test(aktiv_week26 - aktiv_week0, placebo_week26 - placebo_week0)`

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The effect over 26 weeks is to be understood as the difference between week 0 and week 26 in the two groups, ie. $\Delta_{active}$ and $\Delta_{placebo}$. Because it's not the same subjects in the two groups test test is not paired and the correct answer is:

```
t.test(aktiv_week26 - aktiv_week0, placebo_week26 - placebo_week0)
```

this is answer no. 5.

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**Question I.3 (3)**

In order to calculate how many subjects are needed in a new study involving a new group of subjects with medium-high long-term blood sugar, there is a need to determine a confidence interval for the standard deviation of blood sugar reduction among subjects who received the new medicine. Assume that the standard deviation of blood sugar reduction in the sample consisting of subjects who received the new drug is calculated at 0.9.

What is the 95% confidence interval for the standard deviation under these assumptions?

1* ☐ $\left[\sqrt{\frac{(100-1)0.81}{128.42}}, \sqrt{\frac{(100-1)0.81}{73.36}}\right]$

2 ☐ $\left[\sqrt{\frac{(200-1)0.81}{128.42}},\ \sqrt{\frac{(200-1)0.81}{73.36}}\right]$

3 ☐ $\left[\frac{(100-1)0.9^2}{128.42},\ \frac{(100-1)0.9^2}{73.36}\right]$

4 ☐ $\left[\sqrt{\frac{(100-1)0.81}{123.23}},\ \sqrt{\frac{(100-1)0.81}{77.05}}\right]$

5 ☐ $\left[\sqrt{\frac{(200-1)0.9^2}{239.96}},\ \sqrt{\frac{(200-1)0.9^2}{161.83}}\right]$

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The confidence interval for the standard deviation can be calculated by the formula

$$KI_\sigma = \left[\sqrt{\frac{(n-1)s^2}{\chi^2_{1-\alpha/2}}};\sqrt{\frac{(n-1)s^2}{\chi^2_{\alpha/2}}}\right]$$

where the quantiles of the $\chi^2$-distribution is based on $n-1$ degrees of freedom, $s$ is given in the question as 0.9, and $n = 100$. The quantiles of the $\chi^2$-distribution can be found in R by

```
qchisq(c(0.025,0.975),df=99)

## [1]  73.36108 128.42199
```

hence we can write the confidence interval as

$$KI_\sigma = \left[\sqrt{\frac{(100-1)0.9^2}{128.42}};\sqrt{\frac{(100-1)0.9^2}{73.36}}\right]$$

this is answer no. 1.

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**Question I.4 (4)**

Based on the above analysis, the newly developed medicine is now required to be compared with a competitor's medicine in a so-called active-control study, also over 26 weeks. It is expected that subjects, who are randomised to the group receiving the competitor's medicine on average will experience a decrease in the blood sugar by 0.9 units (HbA1c [%]) over 26 weeks. While subjects who are randomised to the group receiving the newly developed medicine will experience the same lowering of blood sugar as in the previous placebo controlled study. It is also assumed that the standard deviation is 1.2.

It was desired to have 95% power to show a significant difference, under the specified assumptions, between the newly developed medicine and the competitors medicine with a 5% level of significance. Based on the following R-code, how many subjects should in total be recruitment for the study?

```
power.t.test(delta=-0.4, sd=1.2, power = 0.95, sig.level = 0.05)

##
##      Two-sample t test power calculation
##
##               n = 234.8696
##           delta = 0.4
##              sd = 1.2
##       sig.level = 0.05
##           power = 0.95
##     alternative = two.sided
##
## NOTE: n is number in *each* group

power.t.test(delta=0.9, sd=1.2, power = 0.95, sig.level = 0.05)

##
##      Two-sample t test power calculation
##
##               n = 47.18603
##           delta = 0.9
##              sd = 1.2
##       sig.level = 0.05
##           power = 0.95
##     alternative = two.sided
##
## NOTE: n is number in *each* group
```

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```r
power.t.test(delta=-0.4, sd=1.2, power = 0.95, sig.level = 0.05, type="paired")

##
##      Paired t test power calculation
##
##               n = 118.8917
##           delta = 0.4
##              sd = 1.2
##       sig.level = 0.05
##           power = 0.95
##     alternative = two.sided
##
## NOTE: n is number of *pairs*, sd is std.dev. of *differences* within pairs
```

1 ☐ 119

2* ☐ 470

3 ☐ 96

4 ☐ 48

5 ☐ 235

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The reduction in the placebo-controlled experiment was 8.5-7.2=1.3 (from the table above), hence the expected difference, between the two medicines, is 0.9-1.3=-0.4 . Hence `delta` should be -0.4, and the standard deviation is given as an assumption to 1.2. Power and significance level are also given in the question at 0.95 and 0.05 respectively. Since it is different subjects in the two groups the analysis should not be paired. Hence the first part of the R-output is the correct one and there should be 235 subjects in each group or a total of 470 subjects. This is answer no. 2.

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**Question I.5 (5)**

Following the active control study, a 95% confidence interval for the estimated treatment effect measured as the long-term blood sugar reduction (HbA1c [%]) after 26 weeks of treatment for the newly developed medicine relative to the competitor's medicine (that is, for the difference between treatment groups) is estimated to $[0.33; 0.82]$.

Judge which of the following statements is a reasonable conclusion on the study:

1 ☐ The newly developed medicine results in a significant reduction in blood sugar after 26 weeks, which with 95% confidence is between 0.33% and 0.82% HbA1c.

2 ☐ There is a significant difference between the two treatments ($p$-value $< 0.05$) and for a potential subject there is a maximum of 5% risk that the newly developed medicine will not reduce blood sugar more than the competitor's medicine.

3 ☐ Any subject, can with 95% confidence, expect that the newly developed medicine reduce their long-term blood sugar (HbA1c [%]) by 0.33% - 0.82% more than if they had taken the competitor's medicine.

4* ☐ There is a significant difference between the two treatments ($p$-value $< 0.05$) and with 95% confidence it can be expected that the newly developed medicine on average will lower the blood sugar (HbA1c [%]) by 0.33% - 0.82% more than the competitor's medicine.

5 ☐ There is 95% probability that the newly developed medicine reduce the blood sugar more than the competitor's medicine.

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We simply have to go through the statements one by one

1 The confidence interval is about the difference in the reduction, not the reduction itself, hence 1 is wrong

2 There is indeed a significant difference between the medicine, however it does not say anything about individual subjects, hence 2 is wrong

3 Again the confidence interval does not state anything about reductions for individual subjects.

4 There is a significant difference, further the confidence interval state what will happen on average (or in mean) over many experiments, hence 4 is correct

5 The confidence interval does not say anything about the probability of the new medicine being more effective, it state in what interval we can expect the average difference to be (with 95% certainty), hence 5 is also wrong.

in conclusion answer no 4 is the correct one.

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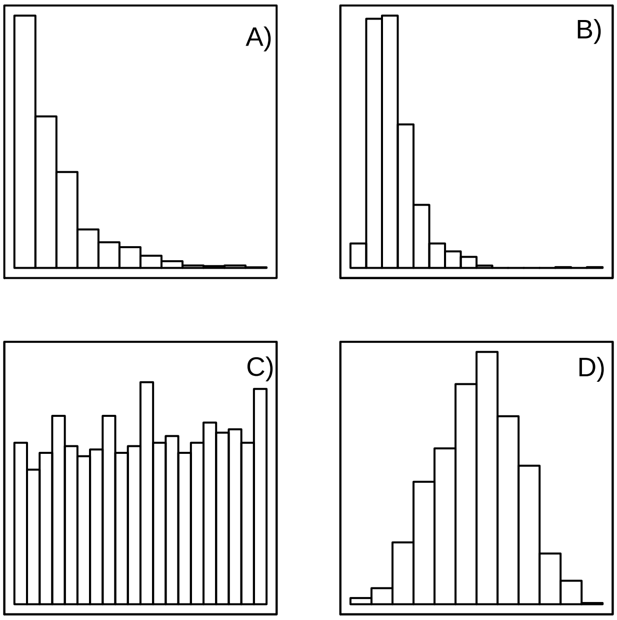
**Exercise II**

The figure below shows histograms of data for realizations from different distributions.




**Question II.1 (6)**

Which of the following statements about the origin of data is probably correct?

1 ☐ B) exponentially distributed data, C) normally distributed data

2 ☐ C) exponentially distributed data, D) normally distributed data

3 ☐ A) exponentially distributed data, D) uniformly distributed data

4 ☐ A) log-normally distributed data, B) normally distributed data

5* ☐ A) exponentially distributed data, B) log-normally distributed data

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We will have to look through the options,

1 B) cannot be exponentially distributed since the highest probability should be near zero, also C) cannot be normally distributed data

2 C) cannot be exponentially distributed either, D) could be normally distributed data

3 A) could be exponentially distributed data, D) cannot be uniformly distributed data

4 A) could be log-normally distributed data (with mean close to zero), however B) cannot be normally distributed data

5 A) could be exponentially distributed data, and B) could also be log-normally distributed data

Hence the only plausible answer i answer no. 5.

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**Exercise III**

The bitterness of wine depends on various factors in the wine making. During the wine making, the grapes are crushed into a mixture of juice and grapes shells and you can, among other things, vary the temperature of the mixture and whether the grape shells are in contact with the juice over a period of time.

In this experiment, 9 evaluators have assessed the bitterness (on a continuous scale) of white wines made under 2 conditions where the contact between shells and juices has been varied (`yes`, `no`). Each assessor has rated 4 wines under each of the 2 conditions. It is assumed that we can ignore the variation between assessors and that all observations are independent.

The following analysis intends to investigate the effect, if any, of contact on the bitterness of the white wines. The following analysis of variance, where some of the numbers have been masked, has been performed:

```
> anova(lm(response ~ contact, data=wine))
Analysis of Variance Table

Response: response
          Df  Sum Sq Mean Sq F value    Pr(>F)
contact    X  3226.7       X    X     0.003479
Residuals 70  X        352.7
```

## Question III.1 (7)

The total sum of squared deviations (SST) is:

1 ☐ 24689

2 ☐ 27563

3 ☐ 3579.4

4* ☐ 27915.7

5 ☐ 2874

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First note that that sum of squared errors (SSE) is $70 \cdot 352.5 = 24689$, and that SST is given by

$$SST = SSE + SSTr = 24675 + 3226.7 = 27915.7$$

which is answer no. 4

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**Question III.2 (8)**

The test statistics and critical value ($\alpha = 0.05$) for the test of whether there are differences in the contact levels are:

1 ☐ The test statistic is 4.57 and the critical value is $F_{0.95}(2, 70) = 3.13$

2* ☐ The test statistic is 9.149 and the critical value is $F_{0.95}(1, 70) = 3.98$

3 ☐ The test statistic is 0.131 and the critical value is $F_{0.95}(1, 70) = 3.98$

4 ☐ The test statistic is 9.149 and the critical value is $F_{0.975}(1, 70) = 5.25$

5 ☐ The test statistic is 6.38 and the critical value is $F_{0.95}(2, 70) = 3.13$

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The test statistic ($F$) is (using that the degrees of freedom for contact is 1)

$$F = \frac{MSTr}{MSE} = \frac{3226.7/1}{352.7} = 9.149$$

this should be compared with a F-distribution with 1 and 70 degrees of freedom, hence we get a critical value of

```
qf(0.95,df1=1,df2=70)

## [1] 3.977779
```

hence the correct answer is 2.

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**Question III.3 (9)**

With the averages of the two levels of contact calculated to $\mu_{contact:no} = 40.52$ and $\mu_{contact:yes} = 53.91$ the 95% confidence interval (post-hoc) for the difference between the two levels of contact is:

1 ☐ $13.39 \pm 1.99\sqrt{24689(\frac{1}{36} + \frac{1}{36})}$

2 ☐ $13.39 \pm 1.96\sqrt{24689(\frac{1}{36} + \frac{1}{36})}$

3 ☐ $13.39 \pm 2.29\sqrt{352.7(\frac{1}{36} + \frac{1}{36})}$

4 ☐ $13.39 \pm 2.29\sqrt{\frac{27915.4}{70}(\frac{1}{36}+\frac{1}{36})}$

5* ☐ $13.39 \pm 1.99\sqrt{352.7(\frac{1}{36}+\frac{1}{36})}$

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First the difference is 53.91-40.52=13.39, hence the post hoc 95% confidence interval is

$$13.39 \pm t_{1-\alpha/2}\sqrt{MSE\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}$$

MSE is given in the anova table as 352.7 and $n_1 = n_2 = 4 \cdot 9 = 36$, and the quantile of the t-distribution should be based on 70 degrees of freedom, ie.

```
qt(0.975,70)

## [1] 1.994437
```

hence the correct answer is answer no. 5.

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## Question III.4 (10)

At significance level $\alpha = 0.05$ what is the conclusion of the study (both conclusion and argument must be correct)?

1* ☐ Contact has a significant influence on the bitterness, since $0.0035 < 0.05$.

2 ☐ Contact has a significant influence on the bitterness, since $3226.7 < 70 \cdot 352.7$.

3 ☐ Contact has a significant influence on the bitterness, since $3226.7 > 352.7$.

4 ☐ Contact does not have a significant influence on the bitterness, since $0.0035 < 0.05$.

5 ☐ Contact does not have a significant influence on the bitterness, since $3226.7 > 352.7$.

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The $p$-value is 0.0035 (directly form the anova table), and hence there is a significant effect, so answer 1-3 have the correct conclusion, further in answer no. 1 the p-value is correctly compared with the significance level, hence 1 is correct. In answer no. 2 MSTr is compared with SSE, this does not tell anything about the significance of the effect. In answer no. 3 MSTr is compared with MSE, which again does not tell anything about the significance of the result. Hence answer no. 1 is the only correct answer.

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**Question III.5 (11)**

It is now reported that "repetitions" at different contact levels is due to different temperature levels. Which of the following statements about a new analysis of variance table that takes into account the different temperature levels is correct (SSQ is short for sum of squares of Q where Q can be residuals or treatment effect)?

1* ☐ MS for contact will remain unchanged, but the $p$-value for contact will change.

2 ☐ SSQ and MS and therefore also the $p$-value for contact will remain unchanged.

3 ☐ SSQ and MS for the residual term will grow but the p-value for contact will remain unchanged.

4 ☐ SSQ and MS for the residual term will remain unchanged.

5 ☐ SSQ and MS for both contact and the residual term will remain unchanged and therefore the p-value for contact will also remain unchanged.

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When an extra level enters into the analysis of variance table the residual sum of squares will be divided between the new effect and the residual term, while the SSQ for the old treatment and therefore also MS for the old treatment will remain unchanged. The change in MSE will result in the another F-statistics for the old effect, and also the degrees of freedom for residuals will change, hence the p-value will change, i.e. answer no. 1 is correct. answer no. 2,3, and 5 state that the p-value will remain unchanged hence these are not correct, 4 is not correct since the residual sum of square will change. Hence the only correct answer is answer no. 1.

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**Exercise IV**

An object is thrown (from the coordinate $(x, y) = (0, 0)$), it is assumed that its path follows the projectile motion without air resistance. The throwing length is given by the expression

$$x_{max} = \frac{v_0^2 \sin(2\theta)}{g}$$

where $g$ is the gravity acceleration, $\theta$ is the throwing angle and $v_0$ is the initial speed. Now suppose that the initial speed and throwing angle are both uncertain, more specifically, that the mean values are $\mu_v$ and $\mu_\theta$ for $v_0$ and $\theta$, respectively, and the variances are $\sigma_v^2$ and $\sigma_\theta^2$. It is further assumed that throwing angle and initial speed are independent, and that $g$ is known without uncertainty. As a help with the task, it is stated that $\frac{\partial \sin(z)}{\partial z} = \cos(z)$.

## Question IV.1 (12)

Which of the following terms is the law of error propagation approximation to the variance of the throwing length?

1* ☐ $\left(\frac{2\mu_v \sin(2\mu_\theta)}{g}\right)^2 \sigma_v^2 + \left(\frac{2\mu_v^2 \cos(2\mu_\theta)}{g}\right)^2 \sigma_\theta^2$

2 ☐ $\left(\frac{2\sigma_v^2 \sin(2\sigma_\theta^2)}{g}\right)^2 \mu_v^2 + \left(\frac{2\sigma_v^2 \cos(2\sigma_\theta^2)}{g}\right)^2 \mu_\theta^2$

3 ☐ $\frac{4\mu_v \cos(2\mu_\theta)}{g}(\sigma_v^2 + \sigma_\theta^2)$

4 ☐ $\left(\frac{4\mu_v \cos(2\mu_\theta)}{g}\right)^2 (\sigma_v^2 + \sigma_\theta^2)$

5 ☐ $\frac{2\sigma_v^2 \sin(2\sigma_\theta^2)}{g}$

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The approximation given by the law of error propagation is given by

$$\sigma_{x_{max}}^2 = \left(\frac{\partial x_{max}}{\partial v_0}\right)^2 \Bigg|_{v_0=\mu_v,\theta=\mu_\theta} \sigma_v^2 + \left(\frac{\partial x_{max}}{\partial \theta}\right)^2 \Bigg|_{v_0=\mu_v,\theta=\mu_\theta} \sigma_\theta^2$$
$$= \left(\frac{2\mu_v \sin(2\mu_\theta)}{g}\right)^2 \sigma_v^2 + \left(\frac{2\mu_v^2 \cos(2\mu_\theta)}{g}\right)^2 \sigma_\theta^2$$

hence the correct answer is answer no. 1

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For specific distribution assumptions, it is possible to construct simulation-based distributions and thus intervals, assuming that the angle (measured in radians) follows a uniform distribution in the range $[\frac{\pi}{8}; \frac{3\pi}{8}]$, and that the initial speed follows a normal distribution with mean 20 and standard deviation $\sigma_v = 2$, and further assumed that $g = 9.81$. As a help to find the interval, the following R-code have been executed (all of which does not necessarily make sense):

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```r
g <- 9.81
v0 <- 20
sigma.v <- 2
k <- 100000
theta <- runif(k, pi/8, pi * 3/8)
v <- rnorm(k, mean=v0, sd=sigma.v)
xm <- v^2 * sin(2 * theta) / g
quantile(xm, prob = c(0.005,0.01,0.025,0.975,0.99,0.995))

##     0.5%       1%     2.5%    97.5%      99%    99.5%
## 19.15289 20.43880 22.53632 54.52955 58.23360 60.63147

mean(xm) + c(-1, 1) * qnorm(0.995) * sd(xm)

## [1] 15.86690 58.26297

mean(xm) + c(-1, 1) * qnorm(0.99)  * sd(xm)

## [1] 17.92003 56.20984

mean(xm) + c(-1, 1) * qnorm(0.975) * sd(xm)

## [1] 20.93523 53.19465
```

## Question IV.2 (13)

Which of the following intervals is the simulation-based interval $(x_1, x_2)$ such that $P(X_{max} < x_1) = P(X_{max} > x_2)$ and $P(x_1 < X_{max} < x_2) = 0.99$?

1* ☐ $(19.15; 60.63)$

2 ☐ $(20.44; 58.23)$

3 ☐ $(17.92; 56.21)$

4 ☐ $(15.87; 58.26)$

5 ☐ $(20.94; 53.19)$

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The vector `xm` consist of `k=100000` realizations of the throwing length under the given assumptions. Hence estimates of the quantile are given directly as quantiles of `xm`. Under the given conditions ($P(X_{max} < x_1) = P(X_{max} > x_2) = 0.005$) we should the the 0.005 and 0.995 quantiles. Hence the correct answer is $(19.15; 60.63)$, which is answer no. 1.

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Exercise V

This exercise is about making a model for predicting CO2 emissions from electricity generation. In power plants materials are burned to generate electricity and in this combustion CO2 is emitted. It is desirable to predict the CO2 level in electricity generation, so that electricity consumption can be moved to periods with the lowest CO2 level and thus minimise emissions.

Every day, CO2 emissions per kWh electricity produced are calculated in Denmark based on data from ENTSO-E (European Network of Transmission System Operators for Electricity).

Data set for determining a good model for predicting CO2 consists of average hourly values of the following variables, the last three being 24 Hour forecasts:

| Variable | Description | Range | Unit |
|---|---|---|---|
| `co2intensity` | CO2 intensity | $[113, 566]$ | gCO2eq/kW |
| `windspeed` | Windspeed | $[1.7, 11.4]$ | m/s |
| `importDE` | Power-import from Germany | $[-2300, 1845]$ | kW |
| `generation` | Generated power | $[920, 3910]$ | kW |

Data is read into R in a data.table `X`.

A linear regression model with an intercept and the three explanatory variables is fitted

$$Y_{\text{co2intensity},i} = \beta_0 + \beta_{\text{w}} X_{\text{windspeed},i} + \beta_{\text{im}} X_{\text{importDE},i} + \beta_{\text{g}} X_{\text{generation},i} + \varepsilon_i, \quad \varepsilon_i \sim N(0, \sigma^2),$$

and the result is:

```
##
## Call:
## lm(formula = "co2intensity ~ windspeed + importDE + generation",
##     data = X)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -90.232 -26.026  -5.613  19.376 153.485
##
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept) 263.146473   8.542834   30.80   <2e-16 ***
## windspeed   -17.202953   0.761038  -22.61   <2e-16 ***
## importDE     -0.031120   0.001192  -26.11   <2e-16 ***
## generation    0.076791   0.002926   26.24   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
```

```
## Residual standard error: 39.73 on 721 degrees of freedom
## Multiple R-squared:  0.7855,Adjusted R-squared:  0.7846
## F-statistic:   880 on 3 and 721 DF,  p-value: < 2.2e-16
```

In the following it is assumed that the assumption about independent errors is fulfilled.

**Question V.1 (14)**

Based on the usual model selection procedure (using significance level $\alpha = 0.05$), should this model be reduced (both conclusion and argument must be correct)?

1 ☐ Yes, since the explained variation is higher than the chosen significance level

2 ☐ No, since the explained variation is higher than the chosen significance level

3 ☐ Yes, since $\hat{\sigma}/df = 39.73/721 = 0.0551$ is higher than the chosen significance level

4 ☐ No, since $\hat{\sigma}/df = 39.73/721 = 0.0551$ is higher than the chosen significance level

5* ☐ No, since all the coefficients in the model are significantly different from zero on the chosen significance level

`------------------------------------ FACIT-BEGIN ------------------------------------`

Lets check the answers: 1 to 4 are really without any meaning, the mentioned values have nothing to do with the significance level. Answer 5 is correct, since according to the backward selection procedure for MLR models no terms (explanatory variables) should be removed from the model, since the $p$-value for each parameter is much below the significance level.

`------------------------------------- FACIT-END -------------------------------------`

**Question V.2 (15)**

Which of the following statements is not correct (the false statement should be identified)?

1 ☐ The model predicts that the CO2-intensity is 417 gCO2eq/kW in conditions with no wind, no power import from Germany and a power generation at 2000 kW per hour

2 ☐ The standard deviation of the errors is estimated to 39.73

3 ☐ The model has explained 79% of the variation

4* ☐ The most important of the three explanatory variables is the wind speed, since $|\hat{\beta}_{\text{im}}| < |\hat{\beta}_{\text{g}}| < |\hat{\beta}_{\text{w}}|$

5 ☐ It is estimated that when the wind speed increase, then the CO2 intensity decrease

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Lets check the answers:

1. A quick calculation gives us $\hat{\beta}_0 + \hat{\beta}_g X_{\text{generation,new}} = 263.14 + 0.076791 \cdot 2000 = 416.72 \approx 417$, since $X_{\text{windspeed,new}} = 0$ and $X_{\text{importDE,new}} = 0$. So this statement is TRUE
2. The estimate of the standard deviation of the errors is found under `Residual standard error:` in the `R` output. So this statement is TRUE
3. The variation explained by the model is found under `Multiple R-squared:` in the `R` output. So this statement is TRUE
4. The value of the coefficient estimates doesn't tell anything about the importance of the explanatory variables. For example if the units of one them was changed (i.e. the values in the data for one of the variables will all be multiplied with the same value) and the same model was applied, then the value of the parameter estimate would change accordingly, but this would not change how much the explanatory variable adds to the explained variation (e.g. the $p$-value would not change). So this statement is FALSE
5. Since the coefficient estimate for wind speed $\hat{\beta}_w$ is negative, it means that when the wind speed increase, then the predicted CO2-intensity will decrease. So this statement is TRUE

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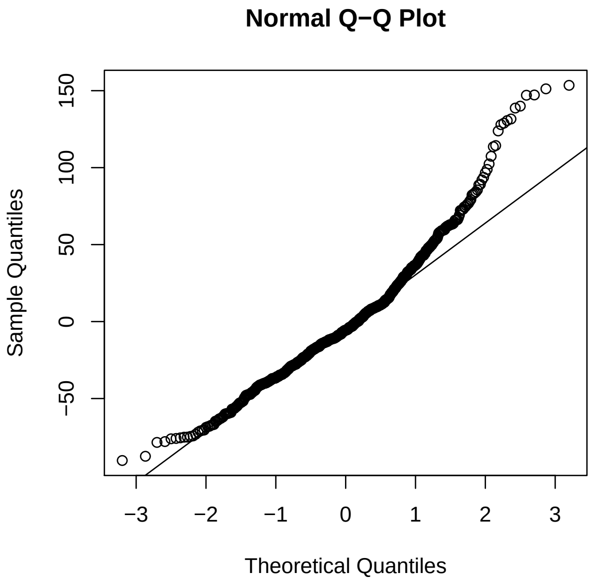
## Question V.3 (16)

In the validation of the model a Q-Q normal plot of the residuals are generated:




Which of the following conclusions can be drawn based on this plot?

1 ☐ The assumption that the errors $\varepsilon_i$ are i.i.d. is not fulfilled

2 ☐ The assumption that the errors $\varepsilon_i$ are i.i.d. is fulfilled

3* ☐ The assumption that the errors $\varepsilon_i$ are normal distributed is not fulfilled

4 ☐ The assumption that the errors $\varepsilon_i$ are normal distributed is fulfilled

5 ☐ It can be rejected that there is a linear relation between the CO2-intensity and the explanatory variables

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The Q-Q normal plot is used to assess if the values (i.e. the residuals, which are the realisations of the errors) are normally distributed. In the case it should be assessed that they are normally distributed, then the points should lay in way which is not clearly systematically deviating from the straight line. In this case, they do clearly deviate from a straight line, so the correct assessment based on the plot is that the errors are not normally distributed. This is answer no. 3

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Continues on page 26

**Question V.4 (17)**

In the matrix formulation of a linear regression model

$$\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}, \quad \boldsymbol{\varepsilon} \sim N(\boldsymbol{0}, \sigma^2\boldsymbol{I}).$$

is the so-called design matrix $\boldsymbol{X}$. In the used model and data set, what are the dimensions of this matrix?

1 ☐ 2884 rows and 4 columns

2* ☐ 725 rows and 4 columns

3 ☐ 4 rows and 4 columns

4 ☐ 5 rows and 4 columns

5 ☐ 4 rows and 5 columns

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The design matrix $\boldsymbol{X}$ is formed by putting the observations of the explanatory variables in a column each and a column of ones for the intercept, so in this case

$$\begin{bmatrix} Y_{\text{co2intensity},i} \\ \vdots \\ Y_{\text{co2intensity},n} \end{bmatrix} = \begin{bmatrix} 1 & x_{\text{windspeed},1} & x_{\text{importDE},1} & x_{\text{generation},1} \\ \vdots & \vdots & & \vdots \\ 1 & x_{\text{windspeed},n} & x_{\text{importDE},n} & x_{\text{generation},n} \end{bmatrix} \begin{bmatrix} \beta_0 \\ \vdots \\ \beta_p \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_n \end{bmatrix}, \quad \varepsilon_i \sim N(0, \sigma^2).$$

and it is seen that the $\boldsymbol{X}$ matrix has $n = df + p + 1 = 721 + 4 = 725$ rows and p+1=4 columns.

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**Question V.5 (18)**

Now the prediction for a new point is wanted. If the new forecasted values of the inputs are in the vector $\boldsymbol{x}_{\text{new}}$, and the model is formulated on matrix form, as in the previous question. How is the prediction of the CO2-intensity calculated?

1* ☐ $\hat{Y}_{\text{co2intensity,new}} = \boldsymbol{x}_{\text{new}}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{Y}$

2 ☐ $\hat{Y}_{\text{co2intensity,new}} = \boldsymbol{x}_{\text{new}}\hat{\boldsymbol{\beta}} + \varepsilon_{\text{new}}$

3 ☐ $\hat{Y}_{\text{co2intensity,new}} = \text{V}(\boldsymbol{x}_{\text{new}}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{Y} + \varepsilon_{\text{new}})$

4 ☐ $\hat{Y}_{\text{co2intensity,new}} = \text{V}(\boldsymbol{x}_{\text{new}}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{Y})$

5 ☐ $\hat{Y}_{\text{co2intensity,new}} = \text{V}(\boldsymbol{x}_{\text{new}}\hat{\boldsymbol{\beta}})$

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The prediction of the model output is

$$\hat{Y}_{\text{co2intensity,new}} = \boldsymbol{x}_{\text{new}}\hat{\boldsymbol{\beta}}$$

and the estimates of the coefficients are

$$\hat{\boldsymbol{\beta}} = (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{Y}$$

which inserted in the previous equation gives the correct prediction (answer no. 1).

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**Exercise VI**

In a production line where lids are produced, one wants to investigate what the probability of errors is. For this purpose, 229 samples have been taken with 770 lids in each sample. The table below shows the result of sampling.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| No. of defective | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| No. of samples | 131 | 38 | 28 | 11 | 4 | 5 | 5 | 2 | 3 | 2 |

Thus, for example, 38 samples (with each 770 lids) have been observed with one defective lid in each. Assuming that the probability of error is the same in each of the 229 samples, the estimate of probability of error can be calculated to

$$\hat{p} = \frac{38 + 2 \cdot 28 + 3 \cdot 11 + 4 \cdot 4 + 5 \cdot 5 + 6 \cdot 5 + 7 \cdot 2 + 8 \cdot 3 + 9 \cdot 2}{770 \cdot 229}$$
$$= 0.00144$$

## Question VI.1 (19)

What is the average number of defective lids per sample?

1 ☐ 4.5

2 ☐ 22.9

3 ☐ 0.33

4* ☐ 1.109

5 ☐ 254

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The average number of defective lids is given by

$$\frac{1}{229}(38 + 2 \cdot 28 + 3 \cdot 11 + 4 \cdot 4 + 5 \cdot 5 + 6 \cdot 5 + 7 \cdot 2 + 8 \cdot 3 + 9 \cdot 2) \quad (1)$$

The fastest way to calculate this number is by noting that it is equal $\hat{p} \cdot 770 = 0.00144 \cdot 770 = 1.109$, which is answer no. 4.

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## Question VI.2 (20)

If `defective` and `samples` denote first and second row, respectively, in the table above and `m` denotes the empirical mean of the number of defective lids per sample. Which of the following

R commands will then calculate the empirical variance for the number of defective lids per sample?

1 ☐ `sum((defective - m)^ 2 * samples)/ (229 * 770)`

2 ☐ `sum((defective - m)^ 2) / 9`

3 ☐ `sqrt(sum((defective - m)^ 2 * samples)/ (229 * 770))`

4* ☐ `sum((defective - m)^ 2 * samples) / 228`

5 ☐ `sum((samples - m)^ 2) / 9`

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The answer here is given by

$$s^2 = \frac{1}{229-1}\sum_{i=1}^{229}(x_i - \bar{x})^2$$
$$= \frac{1}{228}((0-\bar{x})^2 131 + (1-\bar{x})^2 38 + ... + (9-\bar{x})^2 2$$

with $\bar{x}$ =`m`, since `samples`=(131,38,..,2), and `defective`=(0,1,..,9), hence the correct answer is answer no. 4.

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**Question VI.3 (21)**

From which interval will IQR (inter quartile range) for the number of defective lids per sample be calculated?

1 ☐ $[3; 28]$

2 ☐ $[2.25; 6.75]$

3* ☐ $[0; 2]$

4 ☐ $[3.25; 23.75]$

5 ☐ $[2; 7]$

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In order to find the interval we will need the 25% quantile and the 75% quantile. The number of samples are 229 and hence we need the numbers $0.25 \cdot 229 = 57.25$ and $0.75 \cdot 229 = 171.75$, hence the interval is given by observation no. 58 and 172 in the sorted sample. Observation no 58 in the sorted sample is 0, and observation 131+39=170 through 170+28=198 is 2, and hence the interval is $[0, 2]$ which is answer no. 3.

`------------------------------------ FACIT-END ------------------------------------`

**Question VI.4 (22)**

Assuming the probability that a lid is defective is $p = 0.00144$, what is the expected number of samples with 5 defective lids, when there are a total of 229 samples each with 770 lids?

1 ☐ 0.00458

2 ☐ 0.0000225

3 ☐ 0.0173

4* ☐ 1.05

5 ☐ 3.53

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Under the given conditions the number of defective lids in each sample will follow a binomial distribution with $p = 0.00144$ and a sample size of 770. Hence the expected number of samples with 5 defective lids in 229 samples is

$$P(X = 5) \cdot 229 \tag{2}$$

where $X \sim Binom(0.00144, 770)$. This can be calculated in R by

```
dbinom(5,prob=0.00144,770)*229

## [1] 1.048348
```

This is answer no. 4.

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**Question VI.5 (23)**

What is a 95% confidence interval for the probability of defect on a single lid?

1* ☐ $0.00144 \pm z_{0.975}\sqrt{\frac{0.00144(1-0.00144)}{770\cdot 229}} = [0.00126; 0.00162]$

2 ☐ $0.00144 \cdot 229 \pm z_{0.975}\sqrt{\frac{0.00144(1-0.00144)\cdot 229}{770}} = [0.289; 0.370]$

3 ☐ $\frac{229}{770} \pm z_{0.975}\sqrt{\frac{229/770(1-229/770)}{770}} = [0.265; 0.330]$

4 ☐ $0.00144 \cdot \frac{770}{229} \pm z_{0.975}\sqrt{\frac{0.00144(1-0.00144)}{770\cdot 229}} = [0.00466; 0.00502]$

5 ☐ $0.00144 \pm z_{0.975}\frac{\sqrt{229/770(1-229/770)}}{770} = [0.000276; 0.00260]$

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The general formula is

$$\hat{p} \pm z_{1-\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

In our specific case we have $\hat{p} = 0.00144$, $\alpha = 0.05$, and $n = 770 \cdot 229$, which is answer no. 1

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## Question VI.6 (24)

A new sampling is now planned. How many samples with each 770 lids should be taken if you want a margin of error of 0.0001 with a significance level $\alpha = 0.05$ and the observed fraction of defective (0.00144) is used as a scenario?

1 ☐ 384

2 ☐ 2413

3 ☐ 545

4 ☐ 506

5* ☐ 718

```
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```

The number of lid under the specified conditions can be calculated

$$n = p(1-p)\left(\frac{z_{1-\alpha/2}}{ME}\right)^2$$

This is the total number of lids, and the number of samples will therefore be

$$n = \frac{0.00144(1-0.00144)}{770}\left(\frac{z_{1-\alpha/2}}{0.0001}\right)^2$$

in R we can calculate the number by

```r
0.00144 * (1-0.00144) / 770 * (qnorm(0.975)/0.0001)^2

## [1] 717.3682
```

rounding up to the nearest integer gives 718, which is answer no. 5.

```
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```

It is now decided that the distribution assumption is questionable and therefore a simulation-based 95% confidence interval for the defect probability is constructed. For this purpose, the following R code has been executed (all of which does not necessarily make sense) in the code `new.samp` indicates a vector with 131 zeros, 38 ones, etc.

```
k <- 100000
defective <- c(0, 1, 2, 3, 4, 5, 6, 7, 8, 9)
samples <- c(131, 38, 28, 11, 4, 5, 5, 2, 3, 2)
new.samp <- rep(defective, samples)

sim <- replicate(k, sample(new.samp, replace = TRUE))

sim.p <- apply(sim, 2, sum) / (229 * 770)

quantile(sim.p, c(0.025, 0.05, 0.95, 0.975))

##        2.5%          5%         95%       97.5%
## 0.001139908 0.001185278 0.001707027 0.001758067

c(mean(sim.p), sd(sim.p))

## [1] 0.0014406186 0.0001587641
```

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```
##################################################
##
##################################################
sim2 <- replicate(k, sample(samples, replace=TRUE))

sim.p2 <- apply(sim2, 2, sum) / 770

quantile(sim.p2, c(0.025, 0.05, 0.95, 0.975))

##        2.5%          5%        95%      97.5%
## 0.07012987 0.08831169 0.58961039 0.63896104

c(mean(sim.p2), sd(sim.p2))

## [1] 0.2965242 0.1553125
```

**Question VI.7 (25)**

The simulation-based confidence interval becomes:

1* ☐ $[0.00114; 0.00176]$

2 ☐ $[0.0883; 0.590]$

3 ☐ $[0.041; 0.552]$

4 ☐ $[0.0701; 0.639]$

5 ☐ $[0.00119; 0.00171]$

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In order to find the confidence interval we need to sample from the observed distribution, this implemented by the vector `new.samp` hence the distribution of the defect probability is given by `sim.p`, for a 95% confidence interval we need the 0.025 and 0.975 quantiles of `sim.p`. Hence the correct answer is $[0.00114; 0.00176]$, which is answer no 1.

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A similar sample (i.e. 229 samples with 770 lids in each) is now taken from another production line, the result is given in the table below

| No. of defective | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| No. of samples | 73 | 84 | 46 | 17 | 5 | 3 | 1 | 0 | 0 | 0 |

The estimate for the defect probability can be calculated in the same way as above and the result is $\hat{p}_2 = 0.00152$. One now wishes to investigate whether there is a difference in the defect probability in the 2 production lines.

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**Question VI.8 (26)**

What is the usual test statistics and conclusion for the difference in defect probability in the 2 production lines (using the significance level $\alpha = 0.05$)?

1* ☐ $\frac{0.00152-0.00144}{\sqrt{0.00148(1-0.00148)\cdot\frac{2}{229\cdot770}}} = 0.618$, and no difference can documented since $|0.618| < 1.96$

2 ☐ $\frac{0.00152-0.00144}{\sqrt{0.00148(1-0.00148)\cdot\frac{2}{229}}} = 0.0223$, and there is a significant difference since $0.0223 < 0.05$

3 ☐ $\frac{0.00152-0.00144}{\sqrt{0.00148(1-0.00148)\cdot\frac{2}{770}}} = 0.0408$, and there is a significant difference since $0.0408 < 0.05$

4 ☐ $\frac{0.00152-0.00144}{\sqrt{0.00148(1-0.00148)\cdot\left(\frac{1}{229}+\frac{1}{770}\right)}} = 0.0276$, and there is a significant difference since $|0.0276| < 1.96$

5 ☐ $\frac{0.00152-0.00144}{\sqrt{0.00148(1-0.00148)\cdot\frac{2}{229}}} = 0.0223$, and there is a significant difference since $|0.0223| < 1.96$

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In order to calculate the test statistics we will need the estimate of the common defect probability under the null hypothesis, in this case (same number of lids in each sample) it is $\hat{p} = (0.00144 + 0.00152)/2 = 0.00148$, and the test statistic can be calculated by

$$z = \frac{\hat{p}_2 - \hat{p}_1}{\hat{p}(1-\hat{p})\left(\frac{1}{n_1}\frac{1}{n_2}\right)}$$

$$= \frac{0.00152 - 0.00144}{\sqrt{0.00148(1-0.00148)\cdot\left(\frac{1}{229\cdot770} + \frac{1}{229\cdot770}\right)}}$$

$$= \frac{0.00152 - 0.00144}{\sqrt{0.00148(1-0.00148)\cdot\frac{2}{229\cdot770}}}$$

actually the only answer that offer this correct test statistic is answer no. 1, but lets just finish the argument anyway. Since the test is the standard two sided test, the test statistic should be compared with the 0.975 quantile of standard normal distribution, this number equal 1.96. With the test statistics being less than the critical value we cannot document a difference, which is also in line with answer no. 1.

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Regardless of the outcome of the previous questions, it is decided to make a more general comparison of the 2 distributions using tests in contingency tables. For this purpose, the table below has been prepared

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| Defective | line 1 | line 2 |
|---|---|---|
| 0 | 131 | 73 |
| 1 | 38 | 84 |
| 2 | 28 | 46 |
| 3 | 11 | 17 |
| $4-5$ | 9 | 8 |
| $>5$ | 12 | 1 |

**Question VI.9 (27)**

What is the contribution to the usual test statistics for the 4-5 defective group (sum of contributions from both production lines)?

1* ☐ $\frac{1}{17}$

2 ☐ $\frac{9^2}{8.5} + \frac{8^2}{8.5}$

3 ☐ $\frac{2}{17}$

4 ☐ $\frac{8}{8.5} + \frac{9}{8.5}$

5 ☐ $\frac{1}{34}$

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Since the column total is the same (229) in both columns the expected number of defective is $(8+9)/2 = 8.5$, and hence we can calculate the contribution by

$$\frac{(9-8.5)^2}{8.5} + \frac{(8-8.5)^2}{8.5} = \frac{1/4}{17/2} + \frac{1/4}{17/2} = \frac{1}{17}$$

this is answer no 1.

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**Question VI.10 (28)**

The usual test statistic is now calculated to 48.865, what is the p-value and conclusion for the test of the two distributions being equal (use confidence level $\alpha = 0.05$)?

1 ☐ The p-value become $4.31 \cdot 10^{-7}$, and it cannot be rejected that the distributions are equal.

2* ☐ The p-value become $2.36 \cdot 10^{-9}$, and we can reject that the distributions are equal.

3 ☐ The p-value become 0.0201, and it cannot be rejected that the distributions are equal.

4 ☐ The p-value become $4.31 \cdot 10^{-7}$, and it cannot be rejected that the distributions are equal.

5 ☐ The p-value become 0.0201, and we can reject that the distributions are equal.

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The test statistics should be compared with a $\chi^2$-distribution with 5 degrees of freedom, i.e. the p-value is

$$P(X > 48.865)$$

where $X \sim \chi^2(5)$. I R we can calculate this probability by

```
1-pchisq(48.865,df=5)

## [1] 2.364809e-09
```

hence the p-value is $2.36 \cdot 10^{-9}$ and since this is less than the confidence level we can reject the null hypothesis and conclude that the distributions are different, this is answer no 2.

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**Exercise VII**

Let $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$, and assume that $X$ and $Y$ are independent.

**Question VII.1 (29)**

If we assume that $\mu_1 = \mu_2 = 0$, what is $Var(X^2 + Y^2)$ then?

1* ☐ $2\sigma_1^4 + 2\sigma_2^4$

2 ☐ $4\sigma_1^2 + 4\sigma_1^2$

3 ☐ $4\sigma_1^2 + 4\sigma_1^2 + \sigma_1^2\sigma_2^2$

4 ☐ $\sigma_1^4 + \sigma_2^4$

5 ☐ $2\sigma_1^2 + 2\sigma_1^2$

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Since $X$ and $Y$ are independent we can write

$$Var(X^2 + Y^2) = Var(X^2) + Var(Y^2)$$

for $X^2$ we can write (using $E[X] = 0$)

$$Var(X^2) = \sigma_1^4 Var(\frac{X^2}{\sigma_1^2}) = \sigma_1^4 Var(Z)$$

where $Z$ is a random $\chi^2$ distributed random variable with 1 degree of freedom. Hence $Var(Z) = 2$, and $Var(X^2) = 2\sigma^4$, a paralel argument apply for $Var(Y^2)$ and hence the answer is no. 1.

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**Question VII.2 (30)**

If we now assume that $\sigma_1^2 = 2\sigma_2^2$, what is $P((X - \mu_1)^2 > (Y - \mu_2)^2)$ then?

1 ☐ $P(Z > 0)$, where $Z$ follows a $\chi^2$-distribution with 1 degree of freedom.

2* ☐ $P\left(Z > \frac{1}{2}\right)$, where $Z$ follows a $F$-distribution with 1 og 1 degrees of freedom.

3 ☐ $P\left(Z > \frac{1}{2}\right)$, where $Z$ follows a $t$-distribution with 2 degrees of freedom.

4 ☐ $P(Z > 2)$, where $Z$ follows a $\chi^2$-distribution with 1 degree of freedom.

5 ☐ $P\left(|Z| < \frac{1}{2}\right)$, where $Z$ follows a $t$-distribution with 1 degree of freedom.

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We begin by rewritting the expression (using that both sides of the inequality sign are greater than zero)

$$\begin{aligned} P((X-\mu_1)^2 > (Y-\mu_2)^2) =& P\left(\frac{(X-\mu_1)^2}{(Y-\mu_2)^2} > 1\right) \\ =& P\left(\frac{(X-\mu_1)^2/\sigma_1^2}{(Y-\mu_2)^2/\sigma_2^2} > \frac{\sigma_2^2}{\sigma_1^2}\right) \\ =& P\left(\frac{Z_1^2}{Z_2^2} > \frac{\sigma_2^2}{2\sigma_2^2}\right) \\ =& P\left(Z > \frac{1}{2}\right) \end{aligned}$$

since $Z_1$ and $Z_2$ both follow standard normal distributions, we have that $Z_1^2$ and $Z_2^2$ both follow $\chi^2$-distributions with 1 degree of freedom, and in consequence $Z$ will follow a $F$-distribution with 1 and 1 degrees of freedom.

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SÆTTET ER SLUT. God sommer!